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One-loop Quantum Corrections to the Entropy for a 4-dimensional Eternal Black Hole

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Abstract: *The first quantum corrections to the free energy for an eternal 4-dimensional black hole is investigated at one-loop level, in the large mass limit of the black hole, making use of the conformal techniques related to the optical metric. The quadratic and logarithmic divergences as well as a finite part associated with the first quantum correction to the entropy are obtained at a generic temperature. It is argued that, at the Hawking temperature, the horizon divergences of the internal energy should cancel. Some comments on the divergences of the entropy are also presented.*

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Recently, several issues like the interpretation of the Bekenstein-Hawking classical formula for the black hole entropy, the loss of information paradox and the validity of the area law have been discussed in the literature (see, for example, the review [1]). There have also been some attemps to compute semiclassically the first quantum corrections to the Bekenstein-Hawking classical entropy $4\pi GM^2$ [2, 3]. However, so far all the evaluations have been plagued by the appearance of divergences [4, 5, 6, 7, 8, 9, 10] present also in the related "entanglement or geometric entropy" [11, 12, 13].

In the black hole case, the physical origin of these divergences can be traced back to the equivalence principle [14, 15, 16, 17], according to which, in a static space-time with canonical horizons, a system in thermal equilibrium has a local Tolman temperature given by $T(x) = T/\sqrt{-g_{00}(x)}$, T being the generic asymptotic temperature. Since, roughly speaking, very near the horizon a static space-time may be regarded as a Rindler-like space-time, one gets for the Tolman temperature $T(\rho) = T/\rho$, ρ being the distance from the horizon. As a consequence, omitting the multiplicative constant, the total entropy reads

$$S \sim \int d\mathbf{x} \int_{\varepsilon}^{\infty} T^3(\rho)\, d\rho = \frac{AT^3}{2\varepsilon^2}\,, \tag{1}$$

where A is the area of the horizon and ε the horizon cutoff. These considerations suggest that the use of the optical metric $\bar{g}_{\mu\nu} = g_{\mu\nu}/|g_{00}|$, conformally related to the original one, may provide

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an alternative and useful framework to investigate these issues and one purpose of this paper is to implement this idea, which has been proposed in Refs. [18, 19] and put recently forward also in Refs. [20, 21, 10, 22] for the cases of black hole and Rindler space-times.

To start with, we consider a scalar field on a 4-dimensional static space-time defined by the metric (signature $-+++$)

$$ds^2 = g_{00}(\mathbf{x})(dx^0)^2 + g_{ij}(\mathbf{x})dx^i dx^j\,, \qquad \mathbf{x} = \{x^j\}\,, \qquad i,j = 1,...,3\,. \tag{2}$$

The one-loop partition function is given by (we perform the Wick rotation $x_0 = -i\tau$, thus all differential operators one is dealing with will be elliptic)

$$Z = \int d[\phi]\, \exp\left(-\frac{1}{2}\int \phi L_4 \phi d^4x\right)\,, \tag{3}$$

where ϕ is a scalar density of weight $-1/2$ and L_4 is a Laplace-like operator on a 4-dimensional manifold. It has the form

$$L_4 = -\Delta_4 + m^2 + \xi R\,. \tag{4}$$

Here Δ_4 is the Laplace-Beltrami operator, m (the mass) and ξ arbitrary parameters and R the scalar curvature of the manifold.

Now we recall the conformal transformation technique [23, 24, 25, 26]. This method is useful because it permits to compute all physical quantities in an ultrastatic manifold (called the optical manifold [27]) and, at the end of calculations, to transform back them to a static one, with an arbitrary g_{00}. The ultrastatic Euclidean metric $\bar{g}_{\mu\nu}$ is related to the static one by the conformal transformation

$$\bar{g}_{\mu\nu}(\mathbf{x}) = e^{2\sigma(\mathbf{x})} g_{\mu\nu}(\mathbf{x})\,, \tag{5}$$

with $\sigma(\mathbf{x}) = -\frac{1}{2}\ln g_{00}$. In this manner, $\bar{g}_{00} = 1$ and $\bar{g}_{ij} = g_{ij}/g_{00}$ (Euclidean optical metric).

For the one-loop partition function it is possible to show that

$$\bar{Z} = J[g,\bar{g}]\, Z\,, \tag{6}$$

where $J[g,\bar{g}]$ is the Jacobian of the conformal transformation. Such a Jacobian can be explicitely computed [26], but here we shall need only its structural form. Using ζ-function regularization for the determinant of the differential operator we get

$$\ln Z = \ln \bar{Z} - \ln J[g,\bar{g}] = \frac{1}{2}\zeta'(0|\bar{L}_4\ell^2) - \ln J[g,\bar{g}]\,, \tag{7}$$

where ℓ is an arbitrary parameter necessary to adjust the dimensions and ζ' represents the derivative with respect to s of the function $\zeta(s|\bar{L}_4\ell^2)$ related to the operator $\bar{L}_4$, which explicitly reads

$$\bar{L}_4 = e^{-\sigma} L_4 e^{-\sigma} = -\partial_\tau^2 - \bar{\Delta}_3 + \tfrac{1}{6}\bar{R} + e^{-2\sigma}\left[m^2 + (\xi - \tfrac{1}{6})R\right] = -\partial_\tau^2 + \bar{L}_3\,. \tag{8}$$

The same analysis can be easily extended to the finite temperature case [25]. In fact, we recall that for a scalar field in thermal equilibrium at finite temperature $T = 1/\beta$ in an ultrastatic space-time, the partition function $\bar{Z}_\beta$ may be obtained, within the path integral approach, simply by Wick rotation $\tau = ix^0$ and imposing a β periodicity in τ for the field $\bar{\phi}(\tau, x^i)$ [28, 29]. In this way, in the one loop approximation one has

$$\bar{Z}_\beta = \int_{\bar{\phi}(\tau,x^i)=\bar{\phi}(\tau+\beta,x^i)} d[\bar{\phi}]\, \exp\left(-\int_0^\beta d\tau \int \bar{\phi}\bar{L}_4\bar{\phi}\, d^3x\right)\,. \tag{9}$$

Since the Euclidean metric is τ independent, one obtains

$$\begin{aligned}\ln \bar{Z}_\beta &= -\frac{\beta}{2}\left[\mathrm{PP}\,\zeta(-\tfrac{1}{2}|\bar{L}_3) + (2-2\ln 2\ell)\,\mathrm{Res}\,\zeta(-\tfrac{1}{2}|\bar{L}_3)\right] \\ &\qquad + \lim_{s\to 0}\frac{d}{ds}\frac{\beta}{\sqrt{4\pi}\Gamma(s)}\sum_{n=1}^{\infty}\int_0^\infty t^{s-3/2}\,e^{-n^2\beta^2/4t}\ \mathrm{Tr}\,e^{-t\bar{L}_3}\,dt\,, \end{aligned} \tag{10}$$

where PP and Res stand for the principal part and the residue of the function and one has to analytically continue before taking the limit $s \to 0$. As usual, in the definition of ζ-function the subtraction of possible zero-modes of the corresponding operator is left understood. Of course, if the function $\zeta(s|\bar{L}_3)$ is finite for $s = -1/2$, the first term on the right-hand side of the latter equation is just $-\frac{\beta}{2}\zeta(-\frac{1}{2}|\bar{L}_3)$. The latter formula is rigorously valid for a compact manifold. In the paper we shall deal with a non compact manifold, but nevertheless we shall make a formal use of this general formula, employing ζ-function associated with continuum spectrum.

The free energy is related to the canonical partition function by means of the equation

$$F_\beta = E_v + \hat{F}_\beta = -\frac{1}{\beta}\ln Z_\beta = -\frac{1}{\beta}\left(\ln \bar{Z}_\beta - \ln J[g,\bar{g}]\right)\,, \tag{11}$$

where E_v is the vacuum energy while $\hat{F}_\beta$ represents the temperature dependent part (statistical sum). It should be noted that since we are considering a static space-time, the quantity $\ln J[g,\bar{g}]$ depends linearly on β and, according to Eq. (11), it gives contributions only to the vacuum energy term and not to entropy, which may be defined by

$$S_\beta = \beta^2\partial_\beta F_\beta\,, \tag{12}$$

and for the internal energy we assume the well known thermodynamical relation

$$U_\beta = \frac{S_\beta}{\beta} + F_\beta\,. \tag{13}$$

Let us apply this formalism to the case of a massless scalar field in the 4-dimensional Schwarzschild background with a generic gravitational coupling ξR, $\xi = 1/6$ being the conformal coupling. Our aim is to compute the entropy of this field using the latter formula. It may be considered as the prototype of the quantum correction to the classical entropy of a black hole.

To begin with, we recall that the metric is

$$ds^2 = -\left(1-\frac{r_H}{r}\right)(dx^0)^2 + \left(1-\frac{r_H}{r}\right)^{-1} dr^2 + r^2\,d\Omega_2\,, \tag{14}$$

where we are using polar coordinates, r being the radial one and $d\Omega_2$ the 2-dimensional spherical unit metric. The horizon radius is $r_H = 2MG$, M being the mass of the black hole and G the Newton constant. From now on, for the sake of convenience we put $r_H = 1$; in this way all quantities are dimensionless; the dimensions will be easily restored at the end of calculations.

It may be convenient to redefine the Schwarzschild coordinates (x^0, r) by means

$$x'^0 = \frac{x^0}{2}\,, \qquad\qquad \rho = 2(r-1)^{\frac{1}{2}}e^{(r-1)/2}\,. \tag{15}$$

Thus, in the new set of coordinates, we have

$$ds^2 = -4\left(1-\frac{1}{r}\right)(dx'^0)^2 + 4\left(1-\frac{1}{r}\right)\frac{d\rho^2}{\rho^2} + r^2\,d\Omega_2\,, \tag{16}$$

where r is implicitely defined by Eq. (15) and has the expansion, valid near the horizon $r \sim 1$ or $\rho \sim 0$

$$r \sim 1 + \frac{\rho^2}{4} - \frac{\rho^4}{16} + O(\rho^6)\,. \tag{17}$$

The optical metric $\bar{g}_{\mu\nu} = g_{\mu\nu}/(-g_{00})$, which is conformally related to the previous one and appears as an ultrastatic metric, reads

$$d\bar{s}^2 = -(dx'^0)^2 + \frac{1}{\rho^2}\left[d\rho^2 + G(\rho)\,d\Omega_2\right] \tag{18}$$

and has curvature given by

$$\bar{R} = -\frac{6}{r^4} \sim -6 + 6\rho^2 + O(\rho^4)\,, \tag{19}$$

where

$$G(\rho) = r^3 e^{(r-1)} \sim 1 + \rho^2 + O(\rho^4)\,. \tag{20}$$

In order to perform explicit computations, we shall consider the large mass limit of the black hole and this leads to the approximated metric

$$d\bar{s}^2 \sim -(dx'^0)^2 + \frac{1}{\rho^2}\left[d\rho^2 + d\Omega_2\right]\,. \tag{21}$$

This can be considered as an approximation of the metric defined by Eq. (18) in the sense that, near the horizon $\rho = 0$, the geodesics are essentially the same for both the metrics [30]. Eq. (21) defines a manifold with curvature $\bar{R} = -6 + 2\rho^2$. Then, according to Eq. (8), the relevant operator becomes

$$\bar{L}_3 = -\bar{\Delta}_3 - 1 + \left(m^2 + C\right)\rho^2\,, \tag{22}$$

where $\bar{\Delta}_3$ is the related Laplace-Beltrami operator and the constant C acts as an effective mass and has been introduced in order to take into account of the contribution to the curvature (at this order) of the function $G(\rho)$. Working within the first approximation, defined by the metric (21), its value is $C = 1/3$. It should be noted the appearance of an effective "tachionic" mass -1, which has important consequences on the structure of the ζ-function related to the operator $\bar{L}_3$. The invariant measure and the Laplace-Beltrami operator read

$$\begin{aligned} d\bar{V} &= \rho^{-3}\,d\rho\,dV_2\,, \\ \bar{\Delta}_3 &= \rho^2\partial_\rho^2 - \rho\,\partial_\rho + \rho^2\Delta_2\,, \end{aligned} \tag{23}$$

where Δ_2 is the Laplace-Beltrami operator on the unitary sphere S^2 and dV_2 its invariant measure.

In order to study the quantum properties of matter fields defined on this ultrastatic manifold, it is sufficient to investigate the kernel of the operator $e^{-t\bar{L}_3}$. The eigenfunctions of the operator $-\Delta_2 + m^2 + C$ are the spherical harmonics $Y_l^m(\vartheta,\varphi)$ and the eigenvalues $\lambda_l^2 = l(l+1) + m^2 + C$. Let $\Psi_{rlm} = \phi_{rl}(\rho)Y_l^m(\vartheta,\varphi)$ be the eingenfunctions of $\bar{L}_3$ with eigenvalues λ_r^2. The differential equation which determines the continuum spectrum turns out to be

$$\left[\rho^2\,\partial_\rho^2 - \rho\,\partial_\rho - \rho^2\lambda_l^2 + \lambda_r^2 + 1\right]\phi_{rl}(\rho) = 0\,. \tag{24}$$

The only solutions with the correct decay properties at infinity are the Bessel functions of imaginary argument, with $\lambda_r^2 = r^2 \geq 0$. Thus we have

$$\phi_{rl}(\rho) = \rho K_{ir}(\rho\lambda_l)\,. \tag{25}$$

It should be noted that the solutions are vanishing for $\rho = 0$, even if we are not assuming any boundary condition close to the horizon, as in the "brick-wall" regularization of Refs. [4, 6]. As a consequence we have to deal with a continuum spectrum.

For any suitable function $f(\bar{L}_3)$, we may write

$$< x|f(\bar{L}_3)|x >= \int_0^\infty f(r^2) \sum_{lm} \mu_l(r) Y_l^{*m}(\vartheta, \varphi) \phi_{rl}^*(\rho) Y_l^m(\vartheta, \varphi) \phi_{rl}(\rho)\, dr\ , \tag{26}$$

where $\mu_l(r)$ is the spectral measure associated with the continuum spectrum. It is defined by means of equation

$$(\phi_{rl}, \phi_{r'l}) = \frac{\delta(r - r')}{\mu_l(r)}\ . \tag{27}$$

Using the asymptotic behaviour of the Mac Donald functions at the origin [31] one can easily show that $\mu_l(r)$ does not depend on l and reads

$$\mu(r) \equiv \mu_l(r) = \frac{2}{\pi^2} r \sinh \pi r\ . \tag{28}$$

As a consequence, the heat kernel of $\bar{L}_3$ is given by

$$K_t(x|\bar{L}_3) = \int_0^\infty \frac{2}{\pi^2} r\ \sinh \pi r \sum_{l=0}^\infty \frac{(2l+1)}{4\pi} \rho^2 K_{ir}^2(\rho\lambda_l)\, e^{-tr^2}\, dr\ . \tag{29}$$

To go on, we use a method based on the Mellin-Barnes representation for dealing with the sum over l in the latter equation. In fact, for $\mathrm{Re}\, z > 1$ we have [31]

$$\sum_{l=0}^\infty (2l+1)\rho^2 K_{ir}^2(\rho\lambda_l) = \frac{1}{4i\sqrt{\pi}} \int_{\mathrm{Re}\, z>1} \frac{\Gamma(z+ir)\Gamma(z-ir)\Gamma(z)}{\Gamma(z+1/2)}\, \rho^{2-2z}\, f(z)\, dz\ , \tag{30}$$

where

$$f(z) = \sum_{l=0}^\infty (2l+1)\lambda_l^{-2z} \tag{31}$$

is convergent for $\mathrm{Re}\, z > 1$ and may be analytically continued to the whole complex plane using standard methods (see for example [32, 33]). In fact one has

$$f(z) = 2 \sum_{n=0}^\infty \frac{(-1)^n \Gamma(z+n) \left(m^2 + C - \frac{1}{4}\right)^n}{\Gamma(n+1)\Gamma(z)}\, \zeta_H(2z + 2n - 1, 1/2)\ . \tag{32}$$

Here ζ_H represents the Riemann-Hurwitz zeta-function. From the latter equation it follows that $f(z)$ has only a simple pole at $z = 1$ with residue equal to 1 and $f(0) = 1/3 - m^2 - C$. Thus the integrand function in Eq. (30) has simple poles at all the points $z = n$ $(n \leq 1)$ and $z = n \pm ir$ $(n \leq 0)$.

Since we are mainly interested in global quantities, like the partition function, we integrate Eq. (29) over $\bar{\mathcal{M}}^3$, paying attention to the fact that the integration over ρ formally gives rise to divergences. In order to regularize such an integral, we introduce a horizon cutoff $\varepsilon > 0$ for small ρ. When possible, we shall take the limit $\varepsilon \to 0$. In this way we have

$$\begin{aligned} \mathrm{Tr}\, e^{-t\bar{L}_3} = & \frac{A}{(4\pi)^{3/2}} \int_0^\infty dr\, e^{-tr^2} \Bigg[\ r \sinh \pi r \times \\ & \frac{1}{2\pi i} \int_{\mathrm{Re}\, z>1} \frac{\Gamma(z+ir)\Gamma(z-ir)\Gamma(z)}{z\Gamma(z+1/2)}\, \varepsilon^{-2z}\, f(z)\, dz \Bigg]\ , \end{aligned} \tag{33}$$

where the horizon area A is equal to $4\pi r_H^2$. The integrand function in Eq. (33) has the same poles as the function in Eq. (30), with the only exception that $z=0$ is a double pole. Now the advantage in that all the poles in the half plane $\operatorname{Re} z < 0$ give vanishing contributions to the integral in the limit $\varepsilon \to 0$. So we get

$$\begin{aligned} \operatorname{Tr} e^{-t\bar{L}_3} &= \frac{A}{(4\pi)^{3/2}} \left\{ \frac{t^{-3/2}}{2\varepsilon^2} + \left[f(0) \ln \frac{2}{\varepsilon} + \frac{f'(0)}{2} \right] t^{-1/2} \right. \\ &\qquad \left. + \frac{f(0)}{8\sqrt{\pi}} + \frac{f(0)}{\sqrt{\pi}} \int_0^\infty [\psi(ir) + \psi(-ir)]\, e^{-tr^2}\, dr \right\} , \end{aligned} \tag{34}$$

where ψ is the logarithmic derivative of the Γ-function. In the derivation, we made use of

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{-ix}}{x} = i\pi\delta(x) \,. \tag{35}$$

We note that in the t expansion of Eq. (34) appears a term independent of t. Thus, we may write

$$\operatorname{Tr} e^{-t\bar{L}_3} = A \int_0^\infty n(r) e^{-tr^2}\, dr + \frac{Af(0)}{64\pi^2} \tag{36}$$

where

$$n(r) = \frac{r^2}{4\pi^2\varepsilon^2} + \frac{1}{4\pi^2} \left[f(0) \ln \frac{2}{\varepsilon} + \frac{f'(0)}{2} \right] + \frac{f(0)}{8\pi^2} [\psi(ir) + \psi(-ir)] \,. \tag{37}$$

The last term in Eq. (36) may be omitted in the expression of the formal ζ-function, which reads

$$\zeta(s|\bar{L}_3) = A \int_0^\infty n(r) r^{-2s} dr \,. \tag{38}$$

As a consequence, $\zeta(-1/2|\bar{L}_3)$, although formally divergent, is "regular" at $s = -1/2$. This "regularity" is due to the presence of the tachionic mass in the conformally transformed operator $\bar{L}_3$ and its divergence is related to the vacuum state of our quantization scheme in the Schwarzschild coordinates, namely the Boulware vacuum. Formally, the regularity of the quantity $\zeta(-1/2|\bar{L}_3)$ is equivalent to the absence of the conformal anomaly in the optical manifold.

The corresponding partition function may be evaluated making use of Eq. (10), with the replacement $\beta \to \beta/2$ due to the redefinition of the Schwarzschild time in Eq. (15). Finally we have

$$\begin{aligned} \ln Z_\beta &= A\beta j_\varepsilon - \frac{\beta}{4} \zeta(-\tfrac{1}{2}|\bar{L}_3) + \frac{2\pi^2 A}{45\varepsilon^2\beta^3} + \frac{A}{12\beta} \left[f(0) \ln \frac{2}{\varepsilon} + \frac{f'(0)}{2} \right] \\ &\quad - \frac{Af(0)}{64\pi^2} \ln \frac{\beta}{2} - \frac{Af(0)}{8\pi^2} \int_0^\infty \ln\left(1 - e^{-\beta r/2}\right) [\psi(ir) + \psi(-ir)]\, dr \,, \end{aligned} \tag{39}$$

where we have written the Jacobian contribution due to the conformal transformation in the form $A\beta j_\varepsilon$. The horizon divergences which appear in Eq. (39) are also present in the statistical sum contribution to the free energy. The leading one, due to the optical volume, is proportional to the horizon area [4], but in contrast to the Rindler space-time a logarithmic divergence is also present, similar to the one found in Ref. [8, 22].

As is well known, one needs a renormalization in order to remove the vacuum divergences. We recall that these divergenges, as well as the Jacobian conformal factor, being linear in β, do not make contribution to the entropy. However the situation here is complicated by the presence of the horizon divergences, controlled by the cutoff ε. In the Schwarzschild space-time, it is known that the renormalized stress-energy tensor is well defined at the horizon in the

Hartle-Hawking state [15, 19], which in our formalism corresponds to the Hawking temperature $\beta = \beta_H$. This means that the renormalized partition function has to be of the form

$$\begin{aligned}\ln Z^R_\beta &= \frac{A\beta}{90(8\pi)^2\varepsilon^2}\left[\left(\frac{\beta_H}{\beta}\right)^4 + 3\right] - \frac{A\beta}{3(8\pi)^2}\ln\varepsilon\left[\left(\frac{\beta_H}{\beta}\right)^2 + 1\right] \\ &\quad + \frac{A}{12\beta}\left[f(0)\ln 2 + \frac{f'(0)}{2}\right] - \frac{Af(0)}{(8\pi)^2}\ln\frac{\beta}{2} \\ &\quad - \frac{Af(0)}{8\pi^2}\int_0^\infty \ln\left(1 - e^{-\beta r/2}\right)\left[\psi(ir) + \psi(-ir)\right] dr\,, \end{aligned} \tag{40}$$

where we have introduced the Hawking temperature $\beta_H = 4\pi = 8\pi MG$. The first quantum corrections at temperature $T = 1/\beta$ to entropy, free and internal energy immediately follows from Eq. (40). In particular, for the internal energy we have (the dots stay for finite contributions at the horizon, which we do not write down because their value depend on the approximation made)

$$U^R_\beta = \frac{A}{30(8\pi)^2\varepsilon^2}\left[\left(\frac{\beta_H}{\beta}\right)^4 - 1\right] - \frac{A}{3(8\pi)^2}\ln\varepsilon\left[\left(\frac{\beta_H}{\beta}\right)^2 - 1\right] + \dots\,, \tag{41}$$

which has no divergences at $\beta = \beta_H$, while the entropy

$$S_\beta = \frac{8\pi^2 A}{45\varepsilon^2\beta^3} - \frac{A\ln\varepsilon}{6\beta} + \dots \tag{42}$$

also for $\beta = \beta_H$ contains the well known divergent term proportional to the horizon area [4] and, according to Ref. [8], a logarithmic divergence too. Eq. (42) is vanishing in the Boulware vacuum corresponding to $\beta = \infty$.

We conclude with some remarks. In this paper the large mass limit of a 4-dimensional black hole has been investigated making use of conformal transformation techniques, which allow one to work within the so called optical manifold and some advantages have been achieved. We have succeded in obtaining a reasonable expression, valid only for very large black hole mass and near the horizon. The leading contribution gives rise to a divergence of the entropy similar to the one of the Rindler case, but other contributions in general are present, leading to logarithmic divergences and finite parts, which, however, depend on the initial approximation. With regard to this we also would like to mention the results obtained in Ref. [34], where the contributions to the black hole entropy due to modes located inside and near the horizon have been evaluated, using a new invariant statistical mechanical definition for the black hole entropy. The finite contributions, namely the ones indipendent on the horizon cutoff, are compatible with our results.

Finally few words about the horizon divergences. Physically, they may be interpreted in terms of the infinite gravitational redshift existing between the spatial infinity, where one measures the generic equilibrium temperature and the horizon, which is classically inaccessible for the Schwarzschild external observer. With regards to the horizons divergences, we have argued that they are absent in the internal energy at the Hawking temperature. However, they remain in the entropy and in the other thermodynamical quantities, as soon as one assumes the validity of the usual thermodynamical relations. A possible way to deal with such divergences has been suggested in Refs. [4, 34, 20], where it has been argued that the quantum fluctuations at the horizon might provide a natural cutoff. In particular, choosing the horizon cutoff of the order of the Planck length ($\varepsilon^2 = G$), the leading "divergences", evaluated at the Hawking temperature, turns out to be of the form of the the "classical" Bekenstein-Hawking entropy. However one should remark that other terms are present, giving contributions which violate the area law. Alternatively one can try to relate the horizon divergences to the ultaviolet divergences of

quantum gravity, thus arriving at the theory of superstring progagating in a curved space-time [6].

Finally, as far as the Rindler space-time is concerned, this space-time can be treated in a similar way. In our formalism, the massless scalar field in Rindler space-time can be exactly solved, because the optical spatial section turns out to be the hyperbolic space H^3 and the harmonic analysis on such manifold is well known. The computation of the quantum partition function is very similar to the massive black hole space-time and we report it here for comparison. The renormalized free energy may be chosen in the form (here the proper acceleration $a = 1$)

$$F_\beta^R = -\frac{A}{45(8\pi)^2\varepsilon^2}\left[\left(\frac{\beta_U}{\beta}\right)^4 + 3\right] , \tag{43}$$

where now the horizon area A is infinite and $\beta_U = 2\pi a^{-1}$ is the Unruh temperature. As a consequence, the entropy turns out to be

$$S_\beta = \frac{8\pi^2 A}{45\varepsilon^2\beta^3} \tag{44}$$

and this diverges for every finite β, but is zero at zero temperature (the Fulling-Rindler state), which is correct, since we are dealing with a pure state. Furthermore, at $\beta = \beta_U$, corresponding to the Minkowski vacuum, we have a divergent entropy, proportional to the area, regardless of the fact that the Minkowski vacuum is a pure state. This is also to be expected, since an uniformly accelerated observer cannot observe the whole Minkowski space-time. Finally with this renormalization prescription, the internal energy should read

$$U_\beta^R = \frac{A}{15(8\pi)^2\varepsilon^2}\left[\left(\frac{\beta_U}{\beta}\right)^4 - 1\right] \tag{45}$$

and this is vanishing and a fortiori finite at $\beta = \beta_U$, as it should be. Furthermore, at $\beta = \infty$, namely in the Fulling-Rindler vacuum, it is in agreement with the result obtained in Ref. [35].

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